Polygons in buildings and their refined side lengths

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Abstract

As in a symmetric space of noncompact type, one can associate to an oriented geodesic segment in a Euclidean building a vector valued length in the Euclidean Weyl chamber Δ_{euc}. In addition to the metric length it contains information on the direction of the segment. We study in this paper restrictions on the Δ_{euc}-valued side lengths of polygons in Euclidean buildings. The main result is that for thick Euclidean buildings X the set $\mathcal{P}_n(X)$ of possible Δ_{euc}-valued side lengths of oriented n-gons, $n \geq 3$, depends only on the associated *spherical* Coxeter complex. We show moreover that it coincides with the space of Δ_{euc}-valued weights of semistable weighted configurations on the Tits boundary $\partial_{Tits}X$.

The side lengths of polygons in symmetric spaces of noncompact type are studied in the related paper [KLM1]. Applications of the geometric results in both papers to algebraic group theory are given in [KLM3].

Contents

1 Introduction

For a noncompact symmetric space of rank one, such as hyperbolic plane, the only isometry invariant of a geodesic segment is its metric length. In a symmetric space of noncompact type and arbitrary rank the equivalence classes of oriented segments modulo the identity component of the isometry group are parameterized by the Euclidean Weyl chamber Δ_{euc}. We call the vector $\sigma(\gamma) \in \Delta_{euc}$ corresponding to an oriented segment γ its *Δ-length*. The same notion of Δ-length can be defined in a Euclidean building. Note that the directional part of the Δ-length of a segment depends on its orientation. This is because the antipodal involution of the spherical Coxeter complex induces an in general non-trivial involutive self-isometry of the spherical Weyl chamber Δ_{sph}. (Δ_{euc} is the complete Euclidean cone over Δ_{sph}.)

For a Euclidean building or a symmetric space of noncompact type X we denote by $\mathcal{P}_n(X) \subset \Delta^n_{euc}$ the set of Δ-side lengths which occur for oriented n-gons in X.

Main Theorem 1.1. *For a thick Euclidean building X the set $\mathcal{P}_n(X)$ of possible Δ-side lengths of oriented n-gons, $n \geq 3$, depends only on the spherical Weyl chamber Δ_{sph} associated to X.*

In other words, for any two thick Euclidean buildings an isomorphism of *spherical* Coxeter complexes, respectively, an isometry of spherical Weyl chambers induces an isometry of Δ-side length spaces. In particular, automorphisms of Coxeter complexes induce self-isometries.

Our proof of the main theorem uses a relation between polygons in Euclidean buildings and weighted configurations on their spherical Tits buildings at infinity via a Gauss map type construction, see section 4.2. A *weighted configuration* on a spherical building B is a map $\psi : (\mathbb{Z}/n\mathbb{Z}, \nu) \to B$ from a finite measure space. By composing ψ with the natural projection $B \to \Delta_{sph}$ onto the associated spherical Weyl chamber one obtains a map $(\mathbb{Z}/n\mathbb{Z}, \nu) \to \Delta_{sph}$. We call the corresponding point in Δ^n_{euc} the *Δ-weights* of the configuration ψ.

In order to characterize the weighted configurations on $\partial_{Tits}X$ which arise as Gauss maps of polygons in X, we introduce in section 4.1 a notion of *(semi)stability* for weighted configurations on abstract spherical buildings, see also [KLM1, sec. 3.6]. It is motivated by Mumford stability in geometric invariant theory as explained in [KLM1, sec. 4]. If ψ is a weighted configuration on $\partial_{Tits}X$ then one can associate to it a natural convex function on X, the weighted Busemann function b_ψ (well-defined up to an additive constant), and (semi)stability of ψ amounts to certain asymptotic properties of b_ψ.

Theorem 1.2. *Let X be a Euclidean building. Then for $h \in \Delta^n_{euc}$ there exists an oriented n-gon in X with Δ-side lengths h if and only if there exists a semistable weighted configuration on $\partial_{Tits}X$ with Δ-weights h.*

Balser [Bs] proves the sharper result that the weighted configurations on $\partial_{Tits}X$ which arise as Gauss maps of polygons in X are precisely the semistable ones.

Note that every spherical building is the Tits boundary of a Euclidean building, for instance, of the complete Euclidean cone over itself. As a step in our proof of the above results we obtain:

Theorem 1.3. *For a thick spherical building B the set of possible Δ-weights which occur for semistable weighted configurations only depends on the associated spherical Weyl chamber Δ_{sph}.*

In our (logically independent) paper [KLM1] we investigate the Δ-side lengths of polygons in symmetric spaces of noncompact type. We show there that Theorem 1.2 holds also in that case. Theorem 1.3 then implies that Theorem 1.1 holds for symmetric spaces of noncompact type, too.

As a consequence of the results above and in [KLM1] it makes sense to denote by $\mathcal{P}_n(\Delta_{sph}) \subset \Delta_{euc}^n$ the space of Δ-side lengths of oriented n-gons in *thick Euclidean buildings or noncompact symmetric spaces* with spherical Weyl chamber isometric to Δ_{sph}. It coincides with the space of Δ-weights of semistable configurations on thick spherical buildings with this Weyl chamber.

In most cases the spaces $\mathcal{P}_n(\Delta_{sph})$ are known to be *finite-sided convex polyhedral cones*, namely for spherical Coxeter complexes which occur for a symmetric space of noncompact type. As shown in [OSj] and [KLM1] with rather different methods, $\mathcal{P}_n(\Delta_{sph})$ can then be described as the solution set to a finite system of homogeneous linear inequalities. The system can be given explicitely in terms of the Schubert calculus on Grassmann manifolds associated to the symmetric spaces. The case of spherical Coxeter complexes, which occur for thick spherical buildings but not for symmetric spaces of noncompact type, is not covered. (An example for such an exceptional spherical Weyl group is the dihedral group with 16 elements D_8.)

So far we discussed Δ-side lengths. In the case of Euclidean buildings there is a finer invariant for oriented geodesic segments taking values in $E \times E/W_{aff}$ where (E, W_{aff}) denotes the Euclidean Coxeter complex attached to X. We call it the *refined* length. Unlike the Δ-length it also keeps track of the location of the endpoints. In cases when the affine Weyl group acts transitively, for examples for symmetric spaces or ultralimits of thick Euclidean buildings with cocompact affine Weyl group, cf. [KlLe], Δ-length and refined length contain the same information.

An important step in our proof of the Main Theorem is a result concerning refined side lengths, namely the observation that polygons can be transferred between thick Euclidean buildings with isomorphic *Euclidean* Coxeter complexes while keeping their *refined* side lengths fixed, compare Theorem 3.2:

Theorem 1.4. *For a thick Euclidean building X the set $\mathcal{P}_n^{ref}(X) \subset (E \times E/W_{aff})^n$ of possible refined side lengths for n-gons in X depends only on the associated Euclidean Coxeter complex (E, W_{aff}).*

More generally, polygons can be transferred to Euclidean buildings with larger affine Weyl groups while transforming their refined side lengths accordingly (Addendum 3.3), for instance, from a Euclidean building with one vertex to any other thick Euclidean building with isomorphic *spherical* Coxeter complex. Along the way we prove analogous results for polygons in spherical buildings.

The further study of the refined side length spaces $\mathcal{P}_n^{ref}((E, W_{aff}))$ is relevant for certain applications to algebraic group theory and will be taken up in [KLM3].

2 Preliminaries

In this section we will briefly review some basic facts about singular spaces with upper curvature bound, in particular with nonpositive curvature, and about Euclidean and spherical buildings. We will omit most of the proofs. For more details on singular spaces we refer to [Bm, ch. 1-2], [BBI, ch. 4+9], [KlLe, ch. 2] and [Le, ch. 2], and for the theory of buildings from a geometric viewpoint, i.e. within the framework of spaces with curvature bounded above, to [KlLe, ch. 3-4].

2.1 Singular spaces with curvature bounded above

A metric space (Y, d) is called *geodesic* if any two points $x, y \in Y$ can be connected by a distance minimizing geodesic segment, i.e. if there exists an isometric embedding $\sigma : [0, l] \to Y$ such that $d(x, y) = l$, $\sigma(0) = x$ and $\sigma(l) = y$. The image of such a map σ is called a *geodesic segment* connecting x and y and will be denoted by $\overline{xy}$. Note that this is an abuse of notation since, in general, there may be more than one geodesic segment connecting x and y.

Upper curvature bounds. Let Y be a complete geodesic metric space. We do *not* assume that Y is locally compact. One can define *curvature bounds* for such metric spaces by comparison with model spaces of constant curvature. For instance, one can compare the thickness of *geodesic triangles*. Here, by a triangle we mean a one-dimensional object: A triangle in Y with the vertices x, y, z, denoted by $\Delta = \Delta(x, y, z)$, is the union of three geodesic segments $\overline{xy}, \overline{yz}$ and $\overline{zx}$. A *comparison triangle* $\tilde{\Delta} = \Delta(\tilde{x}, \tilde{y}, \tilde{z})$ for Δ in the 2-dimensional model space M^2_k with constant curvature k is a triangle with the same side lengths. To every point p on Δ corresponds a point $\tilde{p}$ on $\tilde{\Delta}$ dividing the corresponding side in the same ratio, and we say that Δ is *thinner* than $\tilde{\Delta}$ if for any points p, q on Δ the *chord comparison* inequality $d(p, q) \leq d(\tilde{p}, \tilde{q})$ holds. The space Y has *curvature* $\leq k$ *(globally)* and is called a *$CAT(k)$-space* if all geodesic triangles with diameter $< 2\,diam(M^2_k)$ are thinner than their comparison triangles in M^2_k. In fact, if $k > 0$, one relaxes the connectivity assumptions and only requires that any two points with distance $< diam(M^2_k)$ are connected by a geodesic segment.

Due to Toponogov's Theorem, a complete simply-connected manifold has curvature $\leq k$ in the distance comparison sense if and only if it has sectional curvature $\leq k$. A metric tree has curvature $-\infty$ in the sense that it has curvature $\leq k$ for all $k \in \mathbb{R}$.

Angles and spaces of directions. The presence of a curvature bound allows to define *angles* between geodesic segments $\sigma_1, \sigma_2 : [0, \epsilon) \to Y$ be unit speed geodesic segments with the same initial point $\rho_1(0) = \rho_2(0) = y$. Let $\tilde{\alpha}(t)$ be the angle of a comparison triangle for $\Delta(y, \sigma_1(t), \sigma_2(t))$ in the appropriate model plane at the vertex corresponding to y. If Y has an upper curvature bound then the comparison angle $\tilde{\alpha}(t)$ is monotonically decreasing as $t \searrow 0$. It therefore converges and we define the angle $\angle_y(\sigma_1, \sigma_2)$ of the segments at y as the limit. In this way, one obtains a pseudo-metric on the space of segments emanating from a point $y \in Y$. The metric space $(\Sigma_y Y, \angle_y)$ obtained by identifying segments with angle zero and metric completion

is called the *space of directions* at p. In the smooth case, $\Sigma_y Y$ is the unit tangent sphere. It turns out that in general $\Sigma_y Y$ is a *CAT(1)-space.*

If $\Delta(x, y, z)$ is a geodesic triangle and $\tilde{\Delta}(\tilde{x}, \tilde{y}, \tilde{z})$ is a comparison triangle, the *angle comparison* $\angle_x(y, z) \leq \angle_{\tilde{x}}(\tilde{y}, \tilde{z})$ holds as a consequence of the definition of angles.

2.2 Hadamard spaces

We will be mainly interested in *CAT(0)-spaces.* These are also called *Hadamard space* since they generalize Hadamard manifolds which are defined to be complete simply-connected Riemannian manifolds of nonpositive curvature. For instance, symmetric spaces of noncompact type are Hadamard manifolds and Euclidean buildings are singular Hadamard spaces.

A basic consequence of the CAT(0)-property is the *convexity* of the distance function, i.e. for any two constant speed geodesic segments $\sigma_1, \sigma_2 : [a, b] \to X$ in a Hadamard space X the distance $t \mapsto d(\sigma_1(t), \sigma_2(t))$ between fellow travellers is a convex function. It follows that any two points can be connected by a unique geodesic segment. In particular, Hadamard spaces are contractible.

Boundary at infinity. A geodesic *ray* is an isometric embedding $\rho : [0, \infty) \to X$. By abusing notation, we will frequently identify geodesic rays with their images. We say that two rays are *asymptotic* if they have bounded Hausdorff distance from each other or, equivalently, if the convex function $t \mapsto d(\rho_1(t), \rho_2(t))$ is bounded and hence nonincreasing. Asymptoticity is an equivalence relation, and the set of equivalence classes of geodesic rays is called the *ideal boundary* or *boundary at infinity* $\partial_\infty X$ of X. An element $\xi \in \partial_\infty X$ is an *ideal* point or a point *at infinity.* A ray representing ξ is said to be *asymptotic to* ξ. We will use the notation $\overline{x\xi}$ to denote the unique geodesic ray from $x \in X$ asymptotic to $\xi \in \partial_\infty X$.

The ideal boundary $\partial_\infty X$ carries a natural topology, called *cone topology*, which will however not play a big role in this paper. A basis for the cone topology is given by subsets of the following form: For a ray $\rho_0 : [0, \infty) \to X$ and numbers $l, \epsilon > 0$ consider all ideal points in $\partial_\infty X$ which are represented by rays $\rho : [0, \infty) \to X$ such that $d(\rho(t), \rho_0(t)) < \epsilon$ for $0 \leq t < l$.

More important for us will be a natural metric on $\partial_\infty X$, the *Tits metric.* Given two ideal points $\xi_1, \xi_2 \in \partial_\infty X$ we pick geodesic rays $\rho_1, \rho_2 : [0, \infty) \to X$ representing them and a point $x \in X$ and, in analogy with the definition of angles above, let $\tilde{\alpha}(t)$ be the angle of a comparison triangle in Euclidean plane at the vertex corresponding to x. The (existence of the) limit $\lim_{t \to \infty} \tilde{\alpha}(t)$ depends only on ξ_1, ξ_2 and not on the location of the initial points $\rho_1(0), \rho_2(0)$ and the base point x. That the limit exists follows from the observation that $\tilde{\alpha}(t)$ increases monotonically as $t \to \infty$ if $\rho_1(0) = \rho_2(0) = x$. We define the *Tits distance* or *Tits angle* $\angle_{Tits}(\xi_1, \xi_2)$ to be this limit. In other words, $2 \sin \frac{\angle_{Tits}(\xi_1, \xi_2)}{2} = \lim_{t \to \infty} \frac{d(\rho_1(t), \rho_2(t))}{t}$. The definition implies the useful inequality

$$\angle_x(\xi, \eta) \leq \angle_{Tits}(\xi, \eta). \tag{1}$$

The metric space $\partial_{Tits} X = (\partial_\infty X, \angle_{Tits})$ is called the *Tits boundary.* As for the spaces of directions, it turns out that the Tits boundary is a CAT(1)-space. Note that the Tits metric does in general *not* induce the cone topology. The Tits metric is lower

semicontinuous with respect to the cone topology and induces a topology which is (in general strictly) finer than the cone topology.

Busemann functions. Busemann functions measure the relative distance from points at infinity. They are constructed as follows. For an ideal point $\xi \in \partial_\infty X$ and a ray $\rho : [0, \infty) \to X$ asymptotic to it we define the *Busemann function* b_ξ as the pointwise monotone limit

$$b_\xi(x) := \lim_{t\to\infty}(d(x, \rho(t)) - t)$$

of normalized distance functions. One checks that, up to an additive constant, b_ξ does not depend on the chosen ray ρ. As a limit of distance functions b_ξ is convex and 1-Lipschitz continuous. The level and sublevel sets of Busemann functions are called horospheres, respectively, horoballs.

Convex functions have directional derivatives. For Busemann functions they are given by the formula

$$\frac{d}{dt^+}(b_\xi \circ \sigma)(t) = -\cos \angle_{\sigma(t)}(\sigma'(t), \xi) \tag{2}$$

where $\sigma : I \to X$ is a unit speed geodesic segment and the angle on the right-hand side is taken between the positive direction $\sigma'(t) \in \Sigma_{\sigma(t)}X$ of the segment σ at $\sigma(t)$ and the ray emanating from $\sigma(t)$ asymptotic to ξ.

Note that along a ray ρ asymptotic to ξ the Busemann function b_ξ is affine linear, i.e. $b_\xi(\rho(t)) = -t + const$. As convex Lipschitz functions Busemann functions are asymptotically linear along any ray ρ and we define the *asymptotic slope* of b_ξ at an ideal point $\eta \in \partial_\infty X$ by

$$slope_\xi(\eta) = \lim_{t\to\infty} \frac{b_\xi(\rho(t))}{t}$$

for a ray ρ asymptotic to η. Since $\angle_{\rho(t)}(\xi, \eta) \nearrow \angle_{Tits}\xi, \eta)$ as $t \to \infty$ one obtains

$$slope_\xi(\eta) = -\cos \angle_{Tits}(\xi, \eta).$$

Cones. Given a metric space with diameter $\leq \pi$ one constructs the *complete Euclidean cone* $Cone(B)$ over B by mimicking the construction which produces Euclidean 3-space from the 2-dimensional unit sphere. The underlying set is $B \times [0, \infty)/\sim$ where $\sim$ collapses $B \times \{0\}$ to a point called the *tip*. For $v_1, v_2 \in B$ and $t_1, t_2 \geq 0$ we consider rays $\rho_i : [0, \infty) \to \mathbb{R}^2$ in Euclidean plane with the same initial point o and angle $\angle_o(\rho_1, \rho_2) = d_B(v_1, v_2)$. We then define the distance of points in $Cone(B)$ represented by (v_1, t_1) and (v_2, t_2) as $d_{\mathbb{R}^2}((\rho_1(t_1), \rho_2(t_2))$.

The space $Cone(B)$ is CAT(0) if and only if B is CAT(1). In this case there is a natural isometry $B \cong \partial_{Tits}Cone(B)$.

2.3 Coxeter complexes

Spherical Coxeter complexes. A *spherical Coxeter complex* (S, W_{sph}) consists of a unit sphere S and a finite subgroup $W_{sph} \subset Isom(S)$ generated by reflections. By a reflection, we mean a reflection at a great sphere of codimension one. W_{sph} is

called the *Weyl group* and the fixed point sets of the reflections in W_{sph} are called *walls*. The pattern of walls gives S a natural structure of a cellular (polysimplicial) complex. The top-dimensional cells, the *chambers*, are fundamental domains for the action $W_{sph} \curvearrowright S$. They are spherical simplices if W_{sph} acts without fixed point. If convenient, we identify the *spherical model Weyl chamber* $\Delta_{sph} = S/W_{sph}$ with one of the chambers in S.

We will only be interested in those spherical Coxeter complexes which are attached to semisimple complex Lie groups, see [Se, chapter V.15] for their classification.

An *embedding* $(S, W_{sph}) \hookrightarrow (S', W'_{sph})$ of spherical Coxeter complexes (of equal dimensions) consists of an isometry $\alpha : S \to S'$ and a compatible monomorphism $\iota : W_{sph} \hookrightarrow W'_{sph}$. The isometry α maps walls to walls. We call the embedding of Coxeter complexes an *isomorphism* if ι is an isomorphism.

Euclidean Coxeter complexes. A *Euclidean Coxeter complex* (E, W_{aff}) consists of a Euclidean space E and a subgroup $W_{aff} \subset Isom(E)$ generated by reflections. Again *reflection* means reflection at a hyperplane. We require moreover that the induced reflection group on the sphere $\partial_{Tits}E$ at infinity is finite.

One obtains an associated spherical Coxeter complex $(\partial_{Tits}E, W_{sph})$. Here $W_{sph} := rot(W_{aff})$ where $rot : Isom(E) \to Isom(\partial_{Tits}E)$ is the natural homomorphism mapping an affine transformation to its linear part. Let Δ_{sph} be the spherical model Weyl chamber of $(\partial_{Tits}E, W_{sph})$. We define the *Euclidean model Weyl chamber* Δ_{euc} of (E, W_{aff}) as the complete Euclidean cone over Δ_{sph}, that is, $\Delta_{euc} = Cone(\Delta_{sph})$. It is canonically identified with the quotient of the vector space of translations on E by the natural action of W_{sph} by conjugation and one has a well-defined addition and scalar multiplication by positive real numbers on Δ_{euc}.

A *wall* in the Coxeter complex (E, W_{aff}) is a hyperplane fixed by a reflection in W_{aff}. *Singular subspaces* are defined as intersections of walls, and *vertices* are zero-dimensional singular subspaces.

We denote the kernel of $rot : W_{aff} \to W_{sph}$ by L_{trans} and we refer to it as the *translation subgroup*. The exact sequence $0 \to L_{trans} \to W_{aff} \to W_{sph} \to 1$ splits, i.e. the affine Weyl group decomposes as the semidirect product $W_{aff} \cong W_{sph} \ltimes L_{trans}$. The two extreme cases are that L is the full group of translations on E, as it happens for the Euclidean Coxeter complex attached to a symmetric space of noncompact type, or that $L = \{0\}$ and $W_{aff} = W_{sph}$ is finite as in the case of Euclidean buildings with one vertex.

A Euclidean Coxeter complex (E, W_{aff}) is called *discrete* if W_{aff} is a discrete subgroup of $Isom(E)$. Discrete Euclidean Coxeter complexes occur as Coxeter complexes attached to Euclidean buildings. If moreover W_{aff} acts cocompactly on E, the pattern of walls induces a natural structure of polysimplicial cell complex on E. The top-dimensional cells, the *alcoves*, are fundamental domains for the action $W_{aff} \curvearrowright E$ and the reflections at the faces of one cell generate the group W_{aff}. The alcoves are canonically isometric to the *model Weyl alcove* E/W_{aff}. The Weyl alcove is different from the Euclidean Weyl chamber.

An *embedding* $(E, W_{aff}) \hookrightarrow (E', W'_{aff})$ of Euclidean Coxeter complexes (of equal dimensions) consists of a homothety $\alpha : E \to E'$ and a compatible monomorphism $\iota : W_{aff} \hookrightarrow W'_{aff}$. The homothety α maps walls to walls. We call the embedding of

Coxeter complexes an *isomorphism* if ι is an isomorphism. A *dilation* of a Coxeter complex (E, W_{aff}) is a self-embedding such that the homothety $\alpha : E \to E$ is a dilation.

2.4 Buildings

Spherical buildings. A *spherical building* modelled on a spherical Coxeter complex (S, W_{sph}) is a CAT(1)-space B together with a maximal atlas of charts, i.e. isometric embeddings $S \hookrightarrow B$. The image of a chart is an *apartment* in B. We require that any two points are contained in a common apartment and that the coordinate changes between charts are induced by isometries in W_{sph}.

We will often denote the metric on a spherical building by $\angle_{Tits}$ because in this paper spherical buildings usually arise as Tits boundaries.

The cell structure and the notions of wall, chamber etc. carry over from the Coxeter complex to the building. The building B is called *thick* if every codimenion-one face is adjacent to at least three chambers. A non-thick building can always be equipped with a natural structure of a thick building by reducing the Weyl group. If W_{sph} acts without fixed points the chambers are spherical simplices and the building carries a natural structure as a piecewise spherical simplicial complex. We will then refer to the cells as simplices.

There is a canonical 1-Lipschitz continuous *accordion* map $acc : B \to \Delta_{sph}$ folding the building onto the model Weyl chamber so that every chamber projects isometrically. $acc(\xi)$ is called the *type* of the point $\xi \in B$, and a point in B is called *regular* if its type is an interior point of Δ_{sph}.

The metric space underlying a 0-dimensional spherical building modelled on the Coxeter complex $(\mathrm{S}^0, \mathbb{Z}_2)$ is a discrete metric space where any two distinct points have distance π.

Euclidean buildings. A *Euclidean building* modelled on a Euclidean Coxeter complex (E, W_{aff}) is a CAT(0)-space X together with a maximal atlas of charts $E \hookrightarrow X$ subject to the following conditions: The charts are isometric embeddings, their images are called *apartments*; any pair of points and, more generally, any ray and any complete geodesic is contained in an apartment; the coordinate changes between charts are restrictions of isometries in W_{aff}.

A Euclidean building is called *thick* if every wall is an intersection of apartments. It is called *discrete* if it is modelled on a discrete Euclidean Coxeter complex. It carries then a natural structure as a polyhedral cell complex.

As an example, the metric space underlying a 1-dimensional Euclidean building modelled on $(\mathbb{R}, W_{aff})$, where $W_{aff} \subset Isom(\mathbb{R})$ is a sungroup generated by reflections at points, is a metric tree. In the discrete case it is a simplicial tree.

If X is a thick Euclidean building modelled on the Coxeter complex (E, W_{aff}) then its Tits boundary $\partial_{Tits}X$ is a thick spherical building modelled on $(\partial_{Tits}E, W_{sph})$. Also the spaces of directions $\Sigma_x X$ are spherical buildings modelled on $(\partial_{Tits}E, W_{sph})$. However, the building $\Sigma_x X$ is thick if and only if x corresponds in a chart to a point in E with maximal possible stabilizer $\cong W_{sph}$.

If B is a spherical building then $Cone(B)$ carries a natural induced Euclidean building structure.

3 Transfer of polygons between buildings

3.1 Polygons and side lengths

By an *n-gon* $z_1 \dots z_n$ in a metric space Z we mean a map $\mathbb{Z}/n\mathbb{Z} \to Z$ carrying i to the *vertex* z_i.

If Z is a CAT(0)-space, such as a Euclidean building or a symmetric space of non-compact type, then any two points in Z are connected by a unique geodesic segment and the polygon can be promoted to a 1-dimensional object. For any pair of successive vertices x_{i-1} and x_i one has a well-defined *side* $\overline{x_{i-1}x_i}$. If Z is a CAT(1)-space, for instance a spherical building, one has well-defined sides for successive vertices of distance $< \pi$. The cyclic ordering of the vertices determines a natural orientation of the sides.

Let (E, W_{aff}) be a *Euclidean* Coxeter complex. To a pair of points (p, q) in E one can associate a vector in the Euclidean Weyl chamber $\Delta_{euc} = Cone(\Delta_{sph})$ as follows. The translations on the affine space E form a vector space on which the spherical Weyl group W_{sph} acts by conjugation. The quotient can be canonically identified with Δ_{euc}. Thus we can attach to (p, q) the image in Δ_{euc} of the translation carrying p to q. We call this vector $\sigma(p, q)$ the *Δ-length* of the oriented geodesic segment $\overline{pq}$. It is invariant under isometries in W_{aff} by construction. Note that the directional part of the Δ-length depends on the orientation of the segment. The reason is that the antipodal involution of the spherical Coxeter complex $(\partial_{Tits}E, W_{sph})$ induces an in general non-trivial involutive self-isometry of the spherical Weyl chamber Δ_{sph}.

The complete invariant of a pair (p, q) modulo the action of W_{aff} is its image $\sigma_{ref}(p, q)$ under the natural projection to $E \times E/W_{aff}$. We call it the *refined length* of the oriented segment $\overline{pq}$. The Δ-length is obtained by composing σ_{ref} with the natural forgetful map $E \times E/W_{aff} \to \Delta_{euc}$. The Δ-length contains the complete information about the metric length and the direction of the segment modulo the spherical Weyl group, while the refined length keeps track in addition of the location of the endpoints. If the affine Weyl group contains the full translation group, as in the case of the Euclidean Coxeter complex attached to a symmetric space of noncompact type, then $E \times E/W_{aff} \cong \Delta_{euc}$ and Δ-length and refined length contain the same information.

As in the Euclidean case, one can attach to a pair of points (p, q) in a *spherical* Coxeter complex (S, W) the *refined length* $\sigma_{ref}(p, q) \in S \times S/W$, and it is invariant under the W-action.

These notions of length carry over to geometries modelled on Coxeter complexes. One chooses an apartment containing a given pair of points and measures length inside the apartment. The length is well-defined because the coordinate changes between apartment charts are restrictions of isometries in the Weyl group.

Hence one has the notion of Δ-length in Euclidean buildings and symmetric spaces of noncompact type, and one has the notion of refined length in Euclidean and spher-

ical buildings. Note that in a symmetric space of noncompact type, although well-defined, the notion of refined length does not give more information than the Δ-length because the affine Weyl group acts transitively.

Let X be a Euclidean building or a symmetric space of noncompact type and let (E, W_{aff}) be its associated Euclidean Coxeter complex. To a polygon $x_1 \dots x_n$ in X we associate its Δ-side lengths $(\sigma(x_0, x_1), \dots, \sigma(x_{n-1}, x_n)) \in \Delta^n_{euc}$ and its refined side lengths $(\sigma_{ref}(x_0, x_1), \dots, \sigma_{ref}(x_{n-1}, x_n)) \in (E \times E/W_{aff})^n$. Analogously one can attach refined side lengths with values in $(S \times S/W)^n$ to n-gons in spherical buildings modelled on the Coxeter complex (S, W).

Definition 3.1. We define $\mathcal{P}_n(X) \subset \Delta^n_{euc}$, respectively $\mathcal{P}^{ref}_n(X) \subset (E \times E/W_{aff})^n$ as the space of possible Δ-side lengths, respectively refined side lengths, which occur for n-gons in X.

3.2 The transfer argument

This section is devoted to the proof of Theorem 1.4 stated in the introduction. In fact we need to prove the same result for spherical buildings since we will proceed by induction on the dimension and apply the induction assumption to the spaces of directions. Recall that the spaces of directions of Euclidean or spherical buildings are spherical buildings, cf. section 2.4.

Transfer Theorem 3.2. *(i) If X and X' are thick Euclidean buildings modelled on the same Euclidean Coxeter complex (E, W_{aff}) then $\mathcal{P}^{ref}_n(X) = \mathcal{P}^{ref}_n(X')$.*

(ii) The analogous assertion for thick spherical buildings modelled on the same spherical Coxeter complex.

In other words, isomorphisms of associated Coxeter complexes (cf. section 2.3) induce bijections of refined side length spaces.

Proof: (ii) We first discuss the spherical case. Let B and B' be thick spherical buildings modelled on the same spherical Coxeter complex (S, W). Given a polygon P in B, we will transfer it to a polygon P' in B' while preserving the refined side lengths. It suffices to show the following assertion: $(*)$ *Let $\xi, \eta, \zeta \in B$ and $\xi', \zeta' \in B'$ so that the oriented segments $\overline{\xi\zeta}$ and $\overline{\xi'\zeta'}$ have equal refined lengths. Then there exists $\eta' \in B'$ so that the triangles $\Delta(\xi, \eta, \zeta)$ and $\Delta(\xi', \eta', \zeta')$ have the same refined side lengths.*

We proceed by induction on the dimension of the buildings. The assertion $(*)$ is trivial in dimension 0. We therefore assume that $dim(B) = dim(B) = d > 0$ and that $(*)$ has been proven in dimensions $< d$.

There is a finite subdivision of the side $\overline{\xi\eta}$ by points $\xi_0 = \xi, \xi_1, ..., \xi_{k-1}, \xi_k = \eta$ such that each geodesic triangle $\Delta(\zeta, \xi_i, \xi_{i+1})$ is contained in an apartment. Namely, choose the subdivision so that each subsegment $\overline{\xi_i\xi_{i+1}}$ is contained in a chamber Δ_i and note that each chamber Δ_i is contained in an apartment through ζ. (The analogous assertion for triangles in Euclidean buildings was proven in [KlLe, Corollary 4.6.8].)

We need to find points $\xi'_1, \ldots, \xi'_k$ in B' such that the triangles $\Delta(\zeta', \xi'_i, \xi'_{i+1})$ have the same refined side lengths as $\Delta(\zeta, \xi_i, \xi_{i+1})$ for all i and such that $\angle_{\xi'_i}(\xi'_{i-1}, \xi'_{i+1}) = \pi$. This will be done by a second induction on i. We can choose ξ'_1 in an apartment containing $\overline{\zeta'\xi'}$. Suppose that ξ'_i has been found, $i \geq 1$. In order to find the direction $\overrightarrow{\xi'_i\xi'_{i+1}}$ at ξ'_i, we apply the induction hypothesis (of the first induction on the dimension) to the links $\Sigma_{\xi_i}B$ and $\Sigma_{\xi'_i}B'$, which are thick spherical buildings of dimension $d-1$ modelled on the same spherical Coxeter complex, and transfer the triangle $\Delta(\overrightarrow{\xi_i\xi_{i-1}}, \overrightarrow{\xi_i\zeta}, \overrightarrow{\xi_i\xi_{i+1}})$ in $\Sigma_{\xi_i}B$ to a triangle $\Delta(\overrightarrow{\xi'_i\xi'_{i-1}}, \overrightarrow{\xi'_i\zeta'}, \overrightarrow{\xi'_i\xi'_{i+1}})$ in $\Sigma_{\xi'_i}B'$ with the same refined side lengths. We then choose an apartment in B' which contains $\overline{\zeta'\xi'_i}$ and is tangent to the direction $\overrightarrow{\xi'_i\xi'_{i+1}}$. Inside this apartment there is a unique choice for ξ'_{i+1} with the desired properties. After transfering all triangles $\Delta(\zeta, \xi_i, \xi_{i+1})$, the concatenation of the segments $\overline{\xi'_i\xi'_{i+1}}$ forms a geodesic segment $\overline{\xi'\eta'}$ with the same refined length as $\overline{\xi\eta}$. This concludes the proof in the spherical case.

(i) The same argument works in the Euclidean case, applying the result for spherical buildings of one dimension less. $\square$

As a consequence, we can define *refined side length spaces* $\mathcal{P}_n^{ref}((E, W_{aff}))$ respectively $\mathcal{P}_n^{ref}((S, W))$ associated to Euclidean and spherical Coxeter complexes. They describe the possible refined side lengths of polygons in thick buildings modelled on these Coxeter complexes. To be consistent with earlier notation we may also write $\mathcal{P}_n^{ref}(\Delta_{sph})$ instead of $\mathcal{P}_n^{ref}((S, W))$.

Our proof of the Transfer Theorem 3.2 allows more generally to transfer polygons from buildings with smaller Weyl groups to buildings with larger Weyl groups. Suppose that

$$(E, W_{aff}) \hookrightarrow (E', W'_{aff}) \tag{3}$$

is an embedding of Euclidean Coxeter complexes, i.e. an isometry $E \to E'$ inducing a monomorphism $W_{aff} \hookrightarrow W'_{aff}$, cf. section 2.3. Our above argument yields:

Addendum 3.3 (to 3.2). *(i) The map*

$$(E \times E/W_{aff})^n \to (E' \times E'/W'_{aff})^n$$

induced by the embedding of Euclidean Coxeter complexes (3) induces a map

$$\mathcal{P}_n^{ref}((E, W_{aff})) \longrightarrow \mathcal{P}_n^{ref}((E', W'_{aff}))$$

of refined side length spaces.

(ii) The analogous assertion for embeddings of spherical Coxeter complexes.

A variation of the transfer construction, namely the folding of polygons into apartments, will be discussed in [KLM3].

4 Polygons and weighted configurations at infinity

The Transfer Theorem 3.2 says that the possible refined side lengths for polygons in a thick Euclidean building depend only on the associated affine Coxeter complex.

We now address our Main Theorem 1.1 and show that the *unrefined* Δ-side lengths depend only on the *spherical* Coxeter complex. That is, we relate the Δ-side lengths of polygons in Euclidean buildings with the same spherical Weyl group but whose affine Weyl groups may have different translation subgroups. Addendum 3.3 allows to transfer polygons from buildings with smaller affine Weyl groups to buildings with larger ones. But to go in the other direction we have to pass through configurations at infinity.

We first introduce in section 4.1 a notion of stability for weighted configurations on spherical buildings which is motivated by (and consistent with, cf. [KLM1, ch. 4]) Mumford stability in geometric invariant theory. In section 4.2 we explain how an oriented polygon in a Euclidean building X gives rise to a collection of Gauss maps which can be regarded as weighted configurations on the spherical Tits building $\partial_{Tits}X$ at infinity, and prove the basic Lemma 4.3 that the arising configurations are semistable. The converse question when a semistable configuration ψ on the Tits boundary $\partial_{Tits}X$ is the Gauss map of a polygon in X amounts to a fixed point problem for a certain weak contraction $\Phi_\psi : X \to X$. In section 4.3 we prove the existence of a fixed point in the special case when X is a Euclidean building with one vertex, i.e. when it is isometric to the complete Euclidean cone over its Tits boundary. In section 4.4 we combine our results and prove the main theorems stated in the introduction.

4.1 Weighted configurations on spherical buildings and stability

Let B be a spherical building. We denote the metric on B by $\angle_{Tits}$ because spherical buildings appear in this paper usually as Tits boundaries.

A collection of points $\xi_1, ..., \xi_n \in B$ and of weights $m_1, ..., m_n \geq 0$ determines a *weighted configuration*

$$\psi : (\mathbb{Z}/n\mathbb{Z}, \nu) \to B$$

on B. Here ν is the measure on $\mathbb{Z}/n\mathbb{Z}$ defined by $\nu(i) = m_i$, and the map ψ sends i to ξ_i. By composing ψ with the natural accordion projection $acc : B \to \Delta_{sph}$ onto the associated spherical Weyl chamber Δ_{sph}, compare section 2.4, one obtains a map $(\mathbb{Z}/n\mathbb{Z}, \nu) \to \Delta_{sph}$. We call the corresponding point $h(\psi) = (h_1, \ldots, h_n)$ in Δ^n_{euc} the Δ-*weights* of the configuration ψ, i.e. $h_i = m_i \cdot acc(\xi_i)$. Recall that Δ_{euc} is defined as the complete Euclidean cone over Δ_{sph}.

The configuration ψ yields, by pushing forward ν, the measure $\mu = \sum m_i \delta_{\xi_i}$ on B. We defined its *slope* function on B by

$$slope_\mu = - \sum_{i \in \mathbb{Z}/n\mathbb{Z}} m_i \cos \angle_{Tits}(\xi_i, \cdot). \tag{4}$$

Definition 4.1 (Stability). The measure μ on B is called *semistable* if $slope_\mu \geq 0$ and *stable* if $slope_\mu > 0$ everywhere on B. The weighted configuration ψ is called (semi)stable if the associated measure has this property.

Example 4.2. Let B be a spherical building of dimension 0. Then a measure μ on B is stable iff all atoms have mass $< \frac{1}{2}|\mu|$, and semistable iff all atoms have mass

$\leq \frac{1}{2}|\mu|$ (and nice semistable iff it it is either stable or consists of two atoms of the same mass).

The terminology *slope* becomes clear when one considers spherical buildings as Tits boundaries. If X is a Euclidean building (or a symmetric space of noncompact type) then we can associate with a measure $\mu = \sum m_i \delta_{\xi_i}$ on $\partial_{Tits} X$ its *weighted Busemann function*

$$b_\mu := \sum_{i \in \mathbb{Z}/n\mathbb{Z}} m_i b_{\xi_i}, \tag{5}$$

on X, cf. the definition of Busemann functions in section 2.2 and the discussion of their asymptotics. The function b_μ is well defined up to an additive constant and *convex*. For any ideal point $\eta \in \partial_{Tits} X$ and any unit speed geodesic ray $\rho : [0, \infty) \to X$ asymptotic to η holds

$$slope_\mu(\eta) = \lim_{t \to \infty} \frac{b_\mu(\rho(t))}{t}, \tag{6}$$

i.e. $slope_\mu(\eta)$ computes the *asymptotic slope* of b_μ in the direction η.

4.2 From polygons to configurations: Gauss maps

Let X be a Euclidean building or a symmetric space of noncompact type. We now relate polygons in X and weighted configurations on the spherical Tits building $\partial_{Tits} X$ at infinity.

Consider a polygon $P = x_1 x_2 \ldots x_n$ in X, i.e. a map $\mathbb{Z}/n\mathbb{Z} \to X$. The distances $m_i = d(x_{i-1}, x_i)$ determine a measure ν on $\mathbb{Z}/n\mathbb{Z}$ by putting $\nu(i) = m_i$. The polygon P gives rise to a collection $Gauss(P)$ of *Gauss maps*

$$\psi : \mathbb{Z}/n\mathbb{Z} \longrightarrow \partial_{Tits} X \tag{7}$$

by assigning to i an ideal point $\xi_i \in \partial_{Tits} X$ so that the ray $\overline{x_{i-1}\xi_i}$ passes through x_i. This construction, in the case of hyperbolic plane, already appears in the letter of Gauss to W. Bolyai [Ga]. Taking into account the measure ν, we view the maps $\psi : (\mathbb{Z}/n\mathbb{Z}, \nu) \to \partial_{Tits} X$ as *weighted configurations* on $\partial_{Tits} X$. Their Δ-weights equal the Δ-side lengths of the polygon P.

Note that if X is a Riemannian symmetric space and the m_i are non-zero, there is a unique Gauss map for P because geodesic segments are uniquely extendable to complete geodesics. On the other hand, if X is a Euclidean building then, due to the branching of geodesics, there are in general several Gauss maps. However, the corresponding weighted configurations have the same Δ-weights.

The following observation is basic for us and explains why the notion of semistability is useful in studying polygons.

Lemma 4.3 (Semistability of Gauss maps). *The pushed forward measures $\mu = \psi_* \nu$ are semistable.*

Proof: Let $\eta \in \partial_{Tits} X$ and let $\gamma_i : [0, m_i] \to X$ be a unit speed parametrization of the geodesic segment $\overline{x_{i-1}x_i}$. Then the Busemann function b_η is one-sided differentiable along γ_i with derivative

$$\frac{d}{dt^+}(b_\eta \circ \gamma_i)(t) = -\cos \angle_{\gamma_i(t)}(\xi_i, \eta) \leq -\cos \angle_{Tits}(\xi_i, \eta),$$

cf. the formula (2) in section 2.2 for the directional derivatives of Busemann functions. Integrating along γ_i we obtain

$$b_\eta(x_i) - b_\eta(x_{i-1}) \le -m_i \cdot \cos \angle_{Tits}(\xi_i, \eta) \tag{8}$$

and summation over all sides yields

$$0 \le -\sum_{i \in \mathbb{Z}/n\mathbb{Z}} m_i \cdot \cos \angle_{Tits}(\xi_i, \eta) = slope_\mu(\eta)$$

confirming the semistability. □

Remark 4.4. If X is a Riemannian symmetric space one can prove the sharper result that the weighted configurations on $\partial_{Tits}X$ arising as Gauss maps of closed polygons in X are *nice* semistable, see [KLM1, Lemma 5.5]. This refinement of the notion of semistability amounts to saying that the associated measures μ are semistable and $\{slope_\mu = 0\}$, if non-empty, is a subbuilding, in fact, the Tits boundary of a totally-geodesic subspace of X, compare [KLM1, Definition 3.12] and the discussion there.

4.3 From configurations to polygons: fixed points for weak contractions

Let X be a Euclidean building or a symmetric space of noncompact type. We are now interested in finding polygons with prescribed Gauss map. Such polygons will correspond to the fixed points of a certain weakly contracting self map of X.

For $\xi \in \partial_{Tits}X$ and $t \ge 0$, we define the map $\phi_{\xi,t} : X \to X$ by sending x to the point at distance t from x on the geodesic ray $\overline{x\xi}$. Since X is nonpositively curved, the function $\delta : t \mapsto d(\phi_{\xi,t}(x), \phi_{\xi,t}(y))$ is convex. It is also bounded because the rays $\overline{x\xi}$ and $\overline{y\xi}$ are asymptotic, and hence it is monotonically non-increasing in t. This means that the maps $\phi_{\xi,t}$ are weakly contracting, i.e. they have Lipschitz constant 1. For a weighted configuration $\psi : (\mathbb{Z}/n\mathbb{Z}, \nu) \to \partial_{Tits}X$ we define the weak contraction

$$\Phi = \Phi_\psi : X \longrightarrow X \tag{9}$$

as the composition $\phi_{\xi_n,m_n} \circ \cdots \circ \phi_{\xi_1,m_1}$. The fixed points of Φ are the n-th vertices of closed polygons $P = x_1 \dots x_n$ with $\psi \in Gauss(P)$.

Regarding the existence of fixed points for Φ, we will only need the special case of buildings with one vertex, that is, of complete Euclidean cones over spherical buildings.

Proposition 4.5. *Suppose that X is a Euclidean building with one vertex and that ψ is a semistable weighted configuration on $\partial_{Tits}X$. Then the weak contraction $\Phi_\psi : X \to X$ has a fixed point.*

The following auxiliary result may be of independent interest. It extends Cartan's fixed point theorem for isometric actions on nonpositively curved spaces with bounded orbits. Note that we do not need to assume local compactness.

Lemma 4.6. *Let Y be a Hadamard space and $\Phi : Y \to Y$ a 1-Lipschitz self map. If the forward orbits $(\Phi^n y)_{n\geq 0}$ are bounded then Φ has a fixed point in Y.*

Proof: Consider an orbit $y_n = \Phi^n y_0$ of a point $y_0 \in Y$ and define the distance from its "tail" by

$$r(y) := \limsup_{n\to\infty} d(y_n, y).$$

Note that r inherits from the distance function the convexity and the 1-Lipschitz continuity. The assumption that Φ is 1-Lipschitz implies

$$r(\Phi y) = \limsup_{n\to\infty} d(y_n, \Phi y) = \limsup_{n\to\infty} d(\Phi y_{n-1}, \Phi y) \leq \limsup_{n\to\infty} d(y_{n-1}, y) = r(y),$$

that is,

$$r \circ \Phi \leq r. \tag{10}$$

It suffices to show that r has a unique minimum since this would then be a fixed point of Φ. We denote

$$\rho := \inf_Y r.$$

For $\epsilon > 0$, let y, y' be points with $r(y) = r(y') < \rho + \epsilon$. Then there exists n_0 such that for $n \geq n_0$ we have

$$d(y_n, y), d(y_n, y') < \rho + \epsilon.$$

On the other hand, let m be the midpoint of $\overline{yy'}$. Since $r(m) \geq \rho$ we have

$$d(y_n, m) > \rho - \epsilon$$

for infinitely many n. For these n we apply triangle comparison to the triangle $\Delta(y, y', y_n)$ with Euclidean plane as model space, cf. section 2.1. For the comparison triangle $\Delta(\tilde{y}, \tilde{y}', \tilde{y}_n)$ in Euclidean plane holds the parallelogram identity:

$$d(\tilde{y}, \tilde{y}')^2 + 4\, d(\tilde{y}_n, \tilde{m})^2 = 2\big(d(\tilde{y}_n, \tilde{y})^2 + d(\tilde{y}_n, \tilde{y}')^2\big)$$

By chord comparison we have $d(y_n, m) \leq d(\tilde{y}_n, \tilde{m})$ and obtain the inequality

$$d(y, y')^2 + 4\,\underbrace{d(y_n, m)}_{>\rho-\epsilon}{}^2 \leq 2\big(\underbrace{d(y_n, y)}_{<\rho+\epsilon}{}^2 + \underbrace{d(y_n, y')}_{<\rho+\epsilon}{}^2\big)$$

and

$$d(y, y')^2 < 16\rho\epsilon + 8\epsilon^2.$$

It follows that any sequence (z_k) in Y with $r(z_k) \searrow \rho$ must be a Cauchy sequence. The completeness of Y implies that r has a minimum, and the minimum must be unique. □

Proof of Proposition 4.5: The building X is isometric to the complete Euclidean cone $Cone(\partial_{Tits} X)$ over its Tits boundary. We denote the unique vertex of X by o.

Due to the conicality of X the contraction Φ has a fairly simple geometry. Let σ be a simplex in $\partial_{Tits} X$ and let V be the corresponding face of X, i.e. the Euclidean sector with tip o and ideal boundary σ. Let $\xi \in \partial_{Tits} X$ and $t \geq 0$. For any face

$W \supseteq V$ of X exists a (maximal) flat $F \subset X$ with $W \subset F$ and $\xi \in \partial_{Tits} F$. The map $\phi_{\xi,t}$ restricts on F to a translation and we have

$$b_\eta(\phi_{\xi,t} x) - b_\eta(x) = -t \cdot \cos \angle_{Tits}(\xi, \eta) \tag{11}$$

for $\eta \in \partial_{Tits} F$ and $x \in F$. Since we may vary W, the equation (11) holds for all $\eta \in \sigma$ and $x \in star(V)$. We define $\check{s}tar(V) := \{x \in star(V) : B_{|\mu|}(x) \subset star(V)\}$ where $|\mu|$ denotes the total mass of $|\mu|$, i.e. $\check{s}tar(V)$ consists of those points in $star(V)$ which have at least distance $|\mu|$ from its boundary. Since Φ has displacement $\leq |\mu|$, we have $\Phi(\check{s}tar(V)) \subset star(V)$ and

$$b_\eta(\Phi x) - b_\eta(x) = slope_\mu(\eta) \tag{12}$$

for $\eta \in \sigma$ and $x \in \check{s}tar(V)$.

We may use the Busemann functions to measure the distance from the vertex o. With the normalization $b_\eta(o) = 0$ we have $d(o, \cdot) = \max_{\eta \in \partial_{Tits} X}(-b_\eta)$. For technical reasons we discretize as follows. We fix a finite subset $F \subset \Delta_{sph}$ in the spherical model Weyl chamber which has the property: If $\eta \in \Delta_{sph}$ and $\zeta \in F$ such that $\angle_{Tits}(\eta, \zeta) \leq 2\angle_{Tits}(\eta, \zeta')$ for all $\zeta' \in F$ then η has distance $> \epsilon$ from all faces of Δ_{sph} which do not contain ζ in their closure. Let $A \subset \partial_{Tits} X$ be the discrete subset consisting of all points with types in F, that is the inverse image of F under the canonical projection $\partial_{Tits} X \to \Delta_{sph}$. By construction A satisfies: If $\eta \in \partial_{Tits} X$ and $\zeta \in A$ such that $\angle_{Tits}(\eta, \zeta) \leq 2\angle_{Tits}(\eta, \zeta')$ for all $\zeta' \in A$ then η has distance $> \epsilon$ from all faces of $\partial_{Tits} X$ which do not contain ζ in their closure, i.e. $B_\epsilon(\eta) \subset star(\sigma_\zeta)$ where σ_ζ denotes the simplex of $\partial_{Tits} X$ containing ζ as an interior point. As an approximation to $d(o, \cdot)$ we use the function

$$f := \max_{\zeta \in A}(-b_\zeta)$$

This function has bounded sublevel sets, and according to Lemma 4.6 we are done once we can show that Φ preserves some non-empty sublevel set.

Let $r > 0$ and $x \in X$ with $d(o, x) > r$. For any $\zeta \in A$, we wish to show that $-b_\zeta(\Phi x) \leq f(x)$. Suppose first that $\angle_o(x, \zeta) \leq 2\angle_o(x, \zeta')$ for all $\zeta' \in A$. Then, if r has been chosen sufficiently large, we have $x \in \check{s}tar(V_\zeta)$ where V_ζ denotes the face of X with ideal boundary σ_ζ. Applying (12) and using that μ is semistable we obtain $-b_\zeta(\Phi x) \leq -b_\zeta(x) \leq f(x)$ in this case. On the other hand, if $\angle_o(x, \zeta) > 2\angle_o(x, \zeta')$ with $\zeta' \in A$ and $\angle_o(x, \zeta') = \min_{\zeta'' \in A} \angle_o(x, \zeta'')$, then $f(x) = -b_{\zeta'}(x) > -b_\zeta(x) + |\mu|$, again if r has been chosen large enough. So $-b_\zeta(\Phi x) \leq -b_\zeta(x) + |\mu| \leq f(x)$ also in this case. We conclude that $f(\Phi x) \leq f(x)$ if $d(o, x)$ is sufficiently large. Since Φ is 1-Lipschitz it follows that it preserves $\{f \leq R\}$ for large enough $R > 0$. □

Remark 4.7. One can show that, if X is a Euclidean building and ψ semistable or if X is a symmetric space of noncompact type and ψ nice semistable, then Φ_ψ has a fixed point. The case of Euclidean buildings which are not necessarily locally compact is due to Andreas Balser [Bs]. The case of symmetric spaces is proven in [KLM1].

4.4 Proofs of the main results

Proof of Theorem 1.2: That the existence of polygons implies the existence of configurations follows from the semistability of Gauss maps, cf. Lemma 4.3.

The converse has been proven in Proposition 4.5 for Euclidean buildings with one vertex. We deduce it for an arbitrary Euclidean building X by using the Addendum 3.3 to the Transfer Theorem 3.2. Namely, suppose that h is the Δ-weight of a semistable configuration on $\partial_{Tits}X$. Then by Proposition 4.5 there exist closed polygons with Δ-side lengths h in the Euclidean building $Cone(\partial_{Tits}X)$. Now the Euclidean Coxeter complex of $Cone(\partial_{Tits}X)$ embeds into the Euclidean Coxeter complex of X; namely if (E, W_{aff}) denotes the Euclidean Coxeter complex attached to X then the Euclidean Coxeter complex attached to $Cone(\partial_{Tits}X)$ is isomorphic to (E, W_{sph}). Addendum 3.3 therefore implies the existence of polygons with Δ-side lengths h in X. $\square$

Notice that our argument does not produce a polygon having the given configuration as a Gauss map, compare remark 4.7.

Proof of Theorem 1.3: Consider two thick spherical buildings B and B' modelled on the same spherical Coxeter complex. The thick Euclidean buildings $Cone(B)$ and $Cone(B')$ are then modelled on the same Euclidean Coxeter complex. The Transfer Theorem 3.2 implies that in both spaces the same refined side lengths occur for closed polygons, in particular also the same Δ-side lengths. It follows from Theorem 1.2 that the same Δ-weights occur for semistable weighted configurations on B and B'. $\square$

Proof of Main Theorem 1.1: Consider two thick Euclidean buildings X and X' and suppose that their spherical Tits buildings $\partial_{Tits}X$ and $\partial_{Tits}X'$ at infinity are modelled on the same spherical Coxeter complex. The spherical buildings are thick as well, and the claim therefore follows from Theorems 1.2 and 1.3. $\square$

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